Berezin Transform

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The Berezin transform associates smooth functions with operators on Hilbert spaces of analytic functions. The usual setting involves an open set $\Omega \subset \mathbf{C}^n$ and a Hilbert space H of analytic functions on Ω. We assume that for each $z \in \Omega$, point evaluation at z is a continuous linear functional on H. Thus for each $z \in \Omega$, there exists $K_z \in H$ such that $f(z) = \langle f, K_z \rangle$ for every $f \in H$. Because K_z reproduces the value of functions in H at z, it is called the *reproducing kernel*. The *normalized reproducing kernel* k_z is defined by $k_z = K_z/\|K_z\|$.

For T a bounded operator on H, the *Berezin transform* of T, denoted $\tilde{T}$, is the complex-valued function on Ω defined by

$$\tilde{T}(z) = \langle T k_z, k_z \rangle.$$

For each bounded operator T on H, the Berezin transform $\tilde{T}$ is a bounded real-analytic function on Ω. Properties of the operator T are often reflected in properties of the Berezin transform $\tilde{T}$. The Berezin transform is named in honor of F. Berezin, who introduced this concept in [4].

The Berezin transform has been useful in several contexts, ranging from the Hardy space (see, for example, [8]) to the Bargmann-Segal space (see, for example, [5]), with major connections to the Bloch space and functions of bounded mean oscillation (see, for example [9]). However, the Berezin transform has been most successful as a tool to study operators on the Bergman space; for concreteness and simplicity, we will restrict attention from now on to that arena.

The Bergman space $L^2_a(D)$ consists of the analytic functions f on the unit disk $D \subset \mathbf{C}$ such that $\int_D |f|^2 \, dA < \infty$ (here dA denotes area measure, normalized so that the area of D equals 1). The normalized reproducing kernel is then given by the formula $k_z(w) = (1 - |z|^2)/(1 - \bar{z}w)^2$.

For $\varphi \in L^\infty(D, dA)$, the *Toeplitz operator* with symbol φ is the operator T_φ on $L^2_a(D)$ defined by $T_\varphi f = P(\varphi f)$, where P is the orthogonal projection of $L^2(D, dA)$ onto $L^2_a(D)$. The Berezin transform of the function φ, denoted $\tilde{\varphi}$, is defined to be the Berezin transform of the Toeplitz operator T_φ. This definition easily leads to the formula

$$\tilde{\varphi}(z) = (1 - |z|^2)^2 \int_D \frac{\varphi(w)}{|1 - \bar{z}w|^4} \, dA(w).$$

If φ is a bounded harmonic function on D, then the mean-value-property can be used to show that $\tilde{\varphi} = \varphi$. The converse was proved by Engliš [6]: if $\varphi \in L^\infty(D, dA)$ and $\tilde{\varphi} = \varphi$, then φ is harmonic on D. Ahern, Flores, and Rudin [1] extended this result to functions $\varphi \in L^1(D, dA)$ (the formula above for $\tilde{\varphi}$ makes sense in this case) and showed that the higher-dimensional analogue is valid up to dimension 11 but fails in dimensions 12 and beyond.

The normalized reproducing kernel k_z tends weakly to 0 as $z \to \partial D$. This implies that if T is a compact operator on the Bergman space L^2_a, then $\tilde{T}(z) \to 0$

as $z \to \partial D$. Unfortunately, the converse fails. For example, if T is the operator on L_a^2 defined $(Tf)(z) = f(-z)$, then $\tilde{T}(z) = (1 - |z|^2)^2/(1 + |z|^2)^2$. Thus in this case $\tilde{T}(z) \to 0$ as $z \to \partial D$, but T is not compact (in fact, this operator T is unitary).

However, the situation is much nicer for Toeplitz operators, and even more generally for finite sums of finite products of Toeplitz operators. Axler and Zheng [2] proved that such an operator is compact if and only if its Berezin transform tends to 0 at ∂D.

The Berezin transform also makes an appearance in the decomposition of the Toeplitz algebra $\mathcal{T}$ generated by the Toeplitz operators with analytic symbol. Specifically, McDonald and Sundberg [7] proved that if $T \in \mathcal{T}$, then T can be written in the form $T = T_\varphi + C$, where φ is in the closed algebra generated by the bounded harmonic functions on the unit disk and C is in the commutator ideal of $\mathcal{T}$. The choice of φ is not unique, but taking φ to be the Berezin transform of T always works (see [3]).

References

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