

Advanced Calculus.; A Formal Background to Mathematics: A Critical Approach to Elementary Analysis.

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The American Mathematical Monthly, Vol. 89, No. 3 (Mar., 1982), 217-219.

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REVIEWS

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COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

Advanced Calculus. By Harold Edwards. Krieger, Huntington, New York, 1980. pp. xvii + 508. ISBN 0-89874-047-9.

A Formal Background to Mathematics: A Critical Approach to Elementary Analysis. By Robert Edwards. 2 vols. Springer-Verlag, New York, 1980. pp. ix + 1170. ISBN 0-387-90513-8.

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Probably every mathematician has had many frustrating conversations with educated non-mathematicians who have no idea what mathematics is or what we do. It is easy to blame this state of affairs on the nature of mathematics, but perhaps we are also partly at fault. Most students who take calculus go no further in mathematics, so the content and style of calculus courses shape the perception of mathematics held by a large percentage of college graduates. Typical calculus courses and typical calculus textbooks, filled with vague and confusing definitions but short on mathematical reasoning and proofs, cannot convey the beauty we see in mathematics. We don't even require calculus students really to *understand* what a limit is, thus

removing from their education one of the key intellectual accomplishments of the nineteenth century.

Those students who still enjoy mathematics after taking calculus usually receive their first exposure to rigorous modern mathematics in a course called either elementary analysis or advanced calculus. Only now do we reveal enough about the structure of the real line so that the student can understand a proof of the Intermediate Value Theorem. The least upper bound axiom, or some other equivalent formulation of the completeness of the real line, is introduced, and finally we can prove that a continuous function on a closed interval attains its maximum. At this level we usually insist that students understand limits, uniform continuity, Riemann integration, etc. These courses often do much more than just correctly cover what could have been done right the first time. The calculus is extended to several variables, where the Fundamental Theorem of Calculus becomes Stokes' Theorem.

Harold Edwards has taken a bold and unusual approach to this subject in his book Advanced Calculus. He has relegated the study of limits, the real line, uniform continuity, etc., to the last chapter, which we would ordinarily expect to be devoted to Stokes' Theorem. He begins immediately with differential forms and gets to Stokes' Theorem very early in the book. The significance of this move is that many students think that the material at the end of the book is less important—after all, courses often don't reach the last chapter of the text. By bringing differential forms from the end-of-the-book ghetto into the bright light of chapter one, the author is shifting the focus of attention.

At first I believed that this approach couldn't be successful. I thought that the formalism of differential forms would be too formidable a structure for students at this level to master. Yet Harold Edwards has made it work. His definitions of differential forms and their integrals seem to be very natural. Plenty of examples are provided and everything is well motivated. For example, the Implicit Function Theorem is stated and proved first for affine maps.

Perhaps it appears that I have contradicted myself by making a statement about one of the key purposes of an advanced calculus course, noting that the book by Harold Edwards does not seem to fulfill it, and then saying that the book works. The crucial point is that students should be introduced to some real mathematics at this level, and what Harold Edwards presents is genuine mathematics, both in spirit and content. The student who has gone through this book will have some grasp of the *nature* of modern mathematics.

For most undergraduates, the insights offered by the standard advanced calculus course are probably more important than the benefits of taming differential forms, so I would be hesitant to recommend Advanced Calculus as the text for a typical undergraduate course. However, this book would be an exciting choice for an honors course or a course for graduate students (unfortunately many entering mathematics graduate students do not know this material). Any mathematician who has been away from differential forms for a while will find that Harold Edwards has provided a pleasant reintroduction. The most important feature of Advanced Calculus is that it is fun—it is fun to read the exercises, it is fun to read the comments printed in the margins, it is fun simply to pick a random spot in the book and begin reading. This is the way mathematics should be presented, with an excitement and liveliness that show why we are interested in the subject.

Robert Edwards (presumably no relation to the Harold Edwards of the first book under review) has written an entirely different type of book. He criticizes many texts for bad lapses from formally correct levels of rigor. These criticisms are often quite valid. It is all too easy to find first year calculus books in which a differential called dy is defined, but with a definition which is just nonsense. Such confusion, usually attributed to an attempt to make things easier for the student, probably hurts the weakest students most.

So Robert Edwards has produced a text which takes a formal approach to elementary analysis. Elementary Analysis (an abbreviation for the full title) intentionally contains few pictures or diagrams because the author believes that "diagrams have no place in a formal approach." The notation is so heavy that the reader will be distracted from the content. For example, having already defined the notion of a sequence converging to 0, we are told that a sequence u converges

to a real number c if $u - c'_N$ converges to 0. Of course, c'_N denotes the obvious constant sequence, but is this notation really going to help anyone understand what's going on or prevent errors in logic?

As another example, having previously shown that $(1 + x/n)^n$ converges as $n \to \infty$ for each real number x, we are given the following definition of the exponential function:

$$\exp \equiv \operatorname{def} \{ z : (\exists x) (\exists y) \big(x \in R \land y \in R \land \big(\text{the sequence } n \leadsto (1 + x/n)^n \text{ converges to the limit } y \big) \}$$
$$\land z = (x, y) \}.$$

And so it goes throughout the entire book.

To be fair, the author assumes that this material is not new to the reader, and that the reader merely wishes to review or to see the formal background which was previously hidden. However, it will be very difficult to use *Elementary Analysis* for review or reference. To understand the notation one must start at the beginning; there is so much cross-referencing that the chapters cannot be read independently. The constant irritation of petty formalism means that the book is no fun to pick up for a few minutes of browsing, and I can't imagine that anyone would enjoy reading the whole book.

Since the beginning of the nineteenth century mathematics has become significantly more formal and more rigorous. This development has given us deeper insight into the intricacies of analysis and has helped steer us away from serious errors. (Actually, it is remarkable how few errors Newton, Euler, etc. made while operating with what we now consider unacceptable sloppiness.) Vague and intuitive notions about infinitesimals have been replaced by precise ε - δ definitions. With nonstandard analysis we can now even remove infinitesimals from the realm of mysticism.

We certainly should not return to the carelessness of eighteenth-century mathematics. But Robert Edwards has inserted far too much formalism into *Elementary Analysis*. The purpose of formalism is to lead to clear and correct thinking, not to obscure the mathematical content. A super-excess of formality at an elementary level only leads to a dull and lifeless presentation. Too much mathematics, especially at the research paper level, is written with the meaning hidden from the reader. *Elementary Calculus* is explicitly aimed at students who will themselves become teachers of calculus. I hope that any future teachers who come across this book will not imitate its style to their students.