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Let $D = \{z \in \mathbf{C}: |z| < 1\}$ be the unit disk in the complex plane and let L^p denote the usual Banach space $L^p = L^p(\partial D, d\theta/2\pi)$. The Hardy space H^p is the subspace of L^p consisting of those functions whose Fourier coefficients corresponding to the negative integers vanish; more precisely,

$$H^p = \left\{g \in L^p: \int_{\partial D} g(z)z^n = 0 \text{ for } n = 1, 2, \cdots\right\}.$$

A function $b \in H^\infty$ is called an inner function if $|b(z)| = 1$ for almost all $z \in \partial D$. The most important class of inner functions are the Blaschke products. A Blaschke product is obtained by taking a sequence $\alpha_1, \alpha_2, \cdots$ in D such that $\sum(1 - |\alpha_n|) < \infty$. An analytic function $b: D \rightarrow \mathbf{C}$ is then defined by

$$b(z) = \prod \frac{|\alpha_n|}{\alpha_n} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}.$$

The summation condition on $\{\alpha_n\}$ insures that the infinite product converges to an analytic function with zeroes at precisely the points $\alpha_1, \alpha_2, \cdots$. By taking radial limits in the usual way, the set of bounded analytic functions defined on D can be identified with the Banach algebra H^∞. Under this identification, the Blaschke product b defined above satisfies $|b(z)| = 1$ for almost all $z \in \partial D$.

Let $C = C(\partial D)$ denote the set of continuous complex-valued functions defined on the circle. It is now well known that the linear span $H^\infty + C$ of H^∞ and C is actually a closed subalgebra of L^∞.

Properties of the algebra $H^\infty + C$ have been useful in several situations. The following theorem expresses the surprising fact that an arbitrary bounded measurable function can be described by two "nice" functions, one from $H^\infty + C$ and the other a Blaschke product.

THEOREM 1. *Let $g \in L^\infty$. Then there exists a Blaschke product b and a function $h \in H^\infty + C$ such that $g = h/b$.*

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To prove this theorem we use a result of R. G. Douglas and W. Rudin [1] which states that $\{h/b: h \in H^\infty \text{ and } b \text{ is a Blaschke product}\}$ is a dense subset of L^∞. Fixing $g \in L^\infty$, let h_n and b_n be such that $\|g - h_n/b_n\|_\infty < 1/n$, where $h_n \in H^\infty$ and b_n is a Blaschke product. The Blaschke product b_n can be written as a product $b_n = c_n d_n$, where c_n is a Blaschke product with a finite number of zeroes and the zeroes

$$\{z_{nj}: j = 1, 2, \cdots\} \text{ of } d_n \text{ satisfy } \textstyle\sum_j \{1 - |z_{nj}|\} < \frac{1}{2^n}\,.$$

Let b be the Blaschke product whose zero set is $\{z_{nj}: n, j = 1, 2, \cdots\}$. Then for each positive integer n,

$$\begin{aligned}\frac{1}{n} &> \|g - h_n/b_n\|_\infty \\ &= \|g - h_n/(c_n d_n)\|_\infty \\ &= \|bg - h_n(b/d_n)(1/c_n)\|_\infty\,.\end{aligned}$$

But $b/d_n \in H^\infty$ and $1/c_n \in C$ and $H^\infty + C$ is an algebra, and so $h_n(b/d_n)(1/c_n) \in H^\infty + C$. Thus $\operatorname{dist}(bg, H^\infty + C) < 1/n$ for each n. Since $H^\infty + C$ is closed we can conclude that $bg \in H^\infty + C$. Letting $h = bg$ gives the desired factorization $g = h/b$.

Since a Blaschke product has absolute value one almost everywhere on the circle, the following corollary is an immediate consequence of Theorem 1.

COROLLARY 1. *Let* $g \in L^\infty$. *Then there exists a function* $h \in H^\infty + C$ *such that* $|g| = |h|$.

The above corollary should be compared with the classical H^∞ statement: if $g \in L^\infty$, then there exists a function $h \in H^\infty$ such that $|g| = |h|$ if and only if $\int \log|g| > -\infty$ or $g = 0$.

If $B \subset L^\infty$ and E is a measurable subset of the circle, then E is called a set of uniqueness for B if 0 is the only function in B which vanishes almost everywhere on E. For example, every set of positive measure is a set of uniqueness for H^∞. In contrast to the H^∞ situation, the next corollary shows that essentially the only set of uniqueness for $H^\infty + C$ is the entire circle.

COROLLARY 2. *Let* E *be a set of uniqueness for* $H^\infty + C$. *Then* $\partial D \sim E$ *has measure zero.*

To prove this corollary, let E be a set of uniqueness for $H^\infty + C$. Let g be the characteristic function of $\partial D \sim E$. By Corollary 1, there is a function $h \in H^\infty + C$ such that $|h| = g$. In particular h vanishes a.e. on E. Since E is a set of uniqueness for $H^\infty + C$, the function h must actually be the zero function. Thus $g = 0$ and so $\partial D \sim E$ has measure zero.

For a compact Hausdorff space X, let $C(X)$ denote the algebra of continuous complex-valued functions defined on X. A closed subalgera B of $C(X)$ is called regular if for every closed set $E \subset X$ and every point $x \in X \sim E$, there is a function $g \in B$ such that $g|E = 0$ and $g(x) \neq 0$. In [2, p. 190] K. Hoffman shows that a regular algebra is not contained in any maximal proper closed subalgebra of $C(X)$.

Because L^∞ is a commutative C^*-algebra it can be identified with $C(M(L^\infty))$; here $M(L^\infty)$ denotes the set of non-zero multiplicative linear functionals on L^∞ with the usual Gelfand topology. Thus H^∞ and $H^\infty + C$ can be thought of as closed subalgebras of $C(M(L^\infty))$ and it makes sense to consider whether they are regular.

Since an H^∞ function cannot vanish on a large set, it is clear that H^∞ is not regular. Again, however, adding on the continuous functions produces a significant change. Corollary 1 implies that $|H^\infty + C| = |L^\infty| = |C(M(L^\infty))|$, where $|B|$ denotes the set $\{|g|: g \in B\}$. Using Urysohn's lemma then gives the following corollary.

COROLLARY 3. *$H^\infty + C$ is a regular subalgebra of L^∞.*

As a consequence of this corollary, $H^\infty + C$ (and thus also H^∞) is not contained in a maximal proper closed subalgebra of L^∞. This fact was previously proved by using fiber algebras, which we now discuss.

Let z denote the identity function on ∂D. For $\alpha \in \partial D$ the fiber X_α of $M(L^\infty)$ over α is the set $X_\alpha = \{\phi \in M(L^\infty): \phi(z) = \alpha\}$. For $g \in L^\infty$, the function $g|X_\alpha$ can be thought of as describing the local behavior of g at the point α.

By restricting the appropriate algebras and functions to a fiber, we obtain local versions of our results. If $g \in C$, then $g|X_\alpha$ is a constant function for each $\alpha \in \partial D$. Thus $(H^\infty + C)|X_\alpha = H^\infty|X_\alpha$ and so in local results we can replace $H^\infty + C$ by H^∞.

K. Hoffman and I.M. Singer [3] proved that $H^\infty|X_\alpha$ is a regular subalgebra of $C(X_\alpha)$. The local version of Corollary 1 shows that much more is true; we see that $|H^\infty|X_\alpha| = |L^\infty|X_\alpha|$.

The local version of Theorem 1 leads to a complete description of the local behavior of an arbitrary L^∞ function.

COROLLARY 4. *Let $g \in L^\infty$ and let $\alpha \in \partial D$. Then there exist Blaschke products b and b_1 and a real-valued function $v \in L^1$ such that*

$$g|X_\alpha = (b\bar{b}_1 \exp[v + i\tilde{v}])|X_\alpha\,,$$

where $\tilde{v}$ denotes the harmonic conjugate of v.

By Theorem 1 and the above remarks, there is an H^∞ function h and a

Blaschke product b_1 such that $g|X_\alpha = (h\bar{b}_1)|X_\alpha$. We can write h as the product of an outer function $\exp[v + i\tilde{v}]$ and an inner function s. The proof of the corollary is now completed by using a result of Kenneth Hoffman which states that if s is an inner function and $\alpha \in \partial D$, then there is a Blaschke product b such that $s|X_\alpha = b|X_\alpha$.

Hoffman's result used above was never published, so a short outline of his proof will be given. For convenience assume that $\alpha = 1$. For $|\lambda| < 1/2$, let

$$s_\lambda(z) = \frac{s(z) + \lambda(1 - z)}{z + \bar{\lambda}(z - 1)s(z)} .$$

Then s_λ is a meromorphic function on D which satisfies $|s_\lambda(z)| = 1$ for almost all $z \in \partial D$. If λ is sufficiently small, then the denominator $z + \bar{\lambda}(z - 1)s(z)$ has precisely one zero on D which is denoted by α_λ. Let

$$b_\lambda(z) = \frac{1 - \bar{\alpha}_\lambda}{1 - \alpha_\lambda} \cdot \frac{z - \alpha_\lambda}{1 - \bar{\alpha}_\lambda z} s_\lambda(z) .$$

Thus b_λ is an inner function in H^∞ and $b_\lambda|X_1 = s|X_1$. An argument of D. J. Newman (see p. 176 of [2]) now shows that for almost all small λ, b_λ is actually a Blaschke product.

If g is a unimodular function in L^∞ and $\alpha \in \partial D$, then in the factorization of Corollary 4 we must have that $v|X_\alpha = 0$, so that in particular v is continuous at α. Thus $g|X_\alpha = (b\bar{b}_1 \exp[i\tilde{v}])|X_\alpha$, where b and b_1 are Blaschke products and v is a real-valued function which is continuous at α. It is of interest to know whether there is a global version of this factorization. To study this question, it is useful to introduce the algebra QC which is defined by $QC = (H^\infty + C) \cap (\overline{H^\infty + C})$; here the bar denotes complex conjugation rather than closure. Functions in QC are called quasi-continuous; it is clear that $C \subset QC$ and it is not hard to see that the inclusion is proper.

Donald Sarason [4] showed that if w is a unimodular function in QC then there are continuous real-valued functions u and v and an integer n such that $w = z^n \exp[i(u + \tilde{v})]$. He also asked the following question, which is still open: Can every unimodular function in $H^\infty + C$ be written as the product of a unimodular function in QC and an inner function?

If Sarason's question has an affirmative answer, then it could be combined with Theorem 1 to show that an arbitrary unimodular function could be written in the form $b\bar{b}_1 \exp[i(u + \tilde{v})]$, where u and v are continuous real-valued functions, b_1 is a Blaschke product, and b is an inner function. To see how this would go, let $g \in L^\infty$ be unimodular. By Theorem 1, there exist $h \in H^\infty + C$ and a Blaschke product b_1 such that $g = h\bar{b}_1$. Since

h is a unimodular function in $H^\infty + C$ it could be written in the form $h = bz^n \exp[i(u + \tilde{v})]$ where u and v are continuous real-valued functions and b is an inner function. Thus $g = b\bar{b}_1 \exp[i(u + \tilde{v})]$ is the desired factorization, where z^n has been absorbed into b if $n > 0$ and into $\bar{b}_1$ if $n < 0$.

Let ψ denote the atom singular inner function $\psi(z) = \exp[(z+1)/(z-1)]$. In [5] Sarason shows that there is a Blaschke product b such that $\psi/b \in C$. It is too much to hope that this will hold for arbitrary singular inner functions. However, it seems to me reasonable to conjecture that if s is a singular inner function, then there exists a Blaschke product b such that $s/b \in QC$. This conjecture seems plausible because Theorem 1 and Corollary 4 do not involve arbitrary inner functions, but only Blaschke products. If this conjecture is true, then Sarason's question could be restated as: If h is a unimodular function in $H^\infty + C$, does there exist a Blaschke product b such that $h/b \in QC$? In any case, it is useful to have a criterion for determining divisibility in QC and $H^\infty + C$. As we will see, the divisibility question can be phrased in terms of Toeplitz operators.

Let P denote the orthogonal projection of L^2 onto H^2. For $g \in L^\infty$, the Toeplitz operator T_g is the operator from H^2 to H^2 defined by $T_g h = P(gh)$.

For b a unimodular function in $H^\infty + C$, let P_b denote the operator $P_b = T_b T_{\bar{b}}$. If b is an inner function, then P_b is the orthogonal projection of H^2 onto $b\, H^2$.

Let $\mathfrak{L}$ denote the set of bounded operators from H^2 to H^2 and let $\mathcal{K} \subset \mathfrak{L}$ denote the compact operators. Then $\mathfrak{L}/\mathcal{K}$ is a C^*-algebra and so it makes sense to talk about projections and ordering in $\mathfrak{L}/\mathcal{K}$. Let $\pi\colon \mathfrak{L} \to \mathfrak{L}/\mathcal{K}$ be the canonical quotient mapping. If $g \in L^\infty$ and $h \in H^\infty + C$ then $\pi(T_{gh}) = \pi(T_g)\pi(T_h)$. Thus $\pi(P_b)$ is a projection if b is a unimodular function in $H^\infty + C$. The next theorem states that divisibility in $H^\infty + C$ corresponds precisely to the ordering in the Calkin algebra.

THEOREM 2. *Let b and w be unimodular functions in $H^\infty + C$. Then $b/w \in H^\infty + C$ if and only if $\pi(P_w) \geqq \pi(P_b)$.*

First suppose that $u = b/w \in H^\infty + C$. Then

$$\begin{aligned}\pi(P_w) - \pi(P_b) &= \pi(T_w T_{\bar{w}}) - \pi(T_{wu} T_{\bar{u}\bar{w}}) \\ &= \pi(T_w)\pi(T_{\bar{w}}) - \pi(T_w)\pi(T_u)\pi(T_{\bar{u}})\pi(T_{\bar{w}}) \\ &= \pi(T_w)[1 - \pi(T_u T_{\bar{u}})]\pi(T_{\bar{w}}) \\ &= \pi(T_w)[1 - \pi(P_u)]\pi(T_w)^* \,.\end{aligned}$$

But $\pi(P_u)$ is a projection and so $1 - \pi(P_u) \geqq 0$. Thus the right-hand side of the above equation is positive and so $\pi(P_w) \geqq \pi(P_b)$.

To prove the implication in the other direction, we introduce Hankel operators. For $g \in L^\infty$, the Hankel operator H_g with symbol g is the operator from H^2 to $L^2 \ominus H^2$ defined by $H_g h = (1 - P)(gh)$. An easy computation shows that $T_{gf} - T_g T_f = H_{\bar{g}}^* H_f$ for $g, f \in L^\infty$.

Now suppose that $\pi(P_w) \geqq \pi(P_b)$. Then

$$0 \leqq \pi(T_w T_{\bar{w}} - T_b T_{\bar{b}})$$

and so

$$\begin{aligned} 0 &\leqq \pi(T_b)^* \pi(T_w T_{\bar{w}} - T_b T_{\bar{b}}) \pi(T_b) \\ &= \pi(T_{\bar{b}w} T_{\bar{w}b} - 1) \\ &= -\pi(H_{b\bar{w}})^* \pi(H_{b\bar{w}}) . \end{aligned}$$

Clearly $\pi(H_{b\bar{w}})^* \ \pi(H_{b\bar{w}})$ is a positive element of the Calkin algebra and the above equation says it is also negative. Thus $\pi(H_{b\bar{w}}{}^* H_{b\bar{w}}) = 0$ which is equivalent to saying that $H_{b\bar{w}}$ is compact. However, if a Hankel operator is compact then its symbol must be in $H^\infty + C$. Thus $b\bar{w} = b/w \in H^\infty + C$, which completes the proof of the theorem.

COROLLARY 5. *Let b and w be unimodular functions in $H^\infty + C$. Then $b/w \in QC$ if and only if $\pi(P_w) = \pi(P_b)$.*

Thus Sarason's question is equivalent to asking whether

$$\{\pi(P_b): b \in H^\infty + C, |b| = 1\} = \{\pi(P_b): b \in H^\infty, |b| = 1\} .$$

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