

The Berezin Transform on the Bergman Space

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reproducing kernel

\mathbb{D} denotes open unit disk in \mathbf{C} .
 dA denotes area measure on \mathbb{D} ,
normalized so that $dA(\mathbb{D}) = 1$.

Definition: *Bergman space*

The *Bergman space* L_a^2 is the set of analytic functions h on \mathbb{D} such that

$$\int_{\mathbb{D}} |h|^2 dA < \infty.$$

With the inner product

$$\langle h, g \rangle = \int_{\mathbb{D}} h \bar{g} dA,$$

the Bergman space L_a^2 is a Hilbert space.

For each $z \in \mathbb{D}$, the point evaluation map

$$h \mapsto h(z)$$

is a bounded linear functional on L_a^2 .

Thus for each $z \in \mathbb{D}$, $\exists k_z \in L_a^2$ such that

$$h(z) = \langle h, k_z \rangle$$

for all $h \in L_a^2$. ($z = \text{zee} = \text{zed}$)

k_z is called the *reproducing kernel* at z for L_a^2 .

If $z, w \in \mathbb{D}$, then

$$k_z(w) = \frac{1}{(1 - \bar{z}w)^2}.$$

Thus if $h \in L_a^2$ and $z \in \mathbb{D}$, then

$$h(z) = \int_{\mathbb{D}} \frac{h(w)}{(1 - z\bar{w})^2} dA(w).$$

Berezin transform

Definition: *normalized reproducing kernel*

For $z \in \mathbb{D}$ the *normalized reproducing kernel* $K_z \in L_a^2$ and is defined by

$$K_z = \frac{k_z}{\|k_z\|_2}.$$

Thus $\|K_z\|_2 = 1$ for all $z \in \mathbb{D}$.

Note that

$$\|k_z\|_2 = \sqrt{\langle k_z, k_z \rangle} = \sqrt{k_z(z)} = \frac{1}{1 - |z|^2}.$$

Thus

$$K_z(w) = \frac{1 - |z|^2}{(1 - \bar{z}w)^2}$$

for all $z, w \in \mathbb{D}$.

Definition: *Berezin transform* [5]

For a bounded linear operator

$$S: L_a^2 \rightarrow L_a^2,$$

the *Berezin transform* of S is the function

$$\tilde{S}: \mathbb{D} \rightarrow \mathbb{C}$$

defined by

$$\tilde{S}(z) = \langle SK_z, K_z \rangle.$$

Note that

$$|\tilde{S}(z)| = |\langle SK_z, K_z \rangle| \leq \|SK_z\|_2 \|K_z\|_2 \leq \|S\|$$

for all $z \in \mathbb{D}$.

properties of the Berezin transform

$$\tilde{S}(z) = \langle SK_z, K_z \rangle$$

If S is a bounded operator on L_a^2 , then

- $|\tilde{S}(z)| \leq \|S\|$ for all $z \in \mathbb{D}$;
- \tilde{S} is a real analytic function on $\mathbb{D} \subset \mathbf{R}^2$;
- $\widetilde{S^*} = \overline{\tilde{S}}$.
- The map

$$S \mapsto \tilde{S}$$

is linear and injective.

Thus \tilde{S} contains all information about S .
What properties of S can be deduced from properties of \tilde{S} ?

Example: S is self-adjoint \iff
 \tilde{S} is a real-valued function.

Claim: $K_z \rightarrow 0$ weakly in L_a^2 as $|z| \uparrow 1$.

In other words, for every $h \in L_a^2$, we have $\langle h, K_z \rangle \rightarrow 0$ as $|z| \uparrow 1$.

Proof: Suppose $h \in L_a^2$ and $\epsilon > 0$. There exists a bounded analytic function f on \mathbb{D} such that $\|h - f\|_2 < \epsilon$. Now

$$\begin{aligned} |\langle h, K_z \rangle| &\leq |\langle h - f, K_z \rangle| + |\langle f, K_z \rangle| \\ &\leq \|h - f\|_2 + (1 - |z|^2)|f(z)| \\ &< \epsilon + (1 - |z|^2)\|f\|_\infty \\ &< 2\epsilon \end{aligned}$$

for $|z|$ sufficiently close to 1. ■

Compact operators send sequences converging weakly to 0 to sequences converging to 0 in norm.

If $S: L_a^2 \rightarrow L_a^2$ is compact, then

$$\tilde{S}(z) \rightarrow 0 \text{ as } |z| \uparrow 1.$$

Proof:

$$|\tilde{S}(z)| = |\langle SK_z, K_z \rangle| \leq \|SK_z\|_2 \rightarrow 0$$

as $|z| \uparrow 1$. ■

Is the converse true? In other words, if $S: L_a^2 \rightarrow L_a^2$ is bounded and

$$\tilde{S}(z) \rightarrow 0 \text{ as } |z| \uparrow 1,$$

then is S compact? No!

Example: Define $S: L_a^2 \rightarrow L_a^2$ by

$$(Sf)(z) = f(-z).$$

Then S is not compact. In fact, S is unitary. However,

$$\tilde{S}(z) = (1 - |z|^2)^2 \langle Sk_z, k_z \rangle$$

$$= (1 - |z|^2)^2 (Sk_z)(z)$$

$$= \frac{(1 - |z|^2)^2}{(1 + |z|^2)^2} \quad \text{because}$$

$$k_z(w) = \frac{1}{(1 - \bar{z}w)^2}$$

$$\rightarrow 0 \text{ as } |z| \uparrow 1.$$

Toeplitz operators

Let $P: L^2(\mathbb{D}) \rightarrow L_a^2$ be the orthogonal projection of $L^2(\mathbb{D})$ onto L_a^2 .

If $u \in L^2(\mathbb{D})$ and $z \in \mathbb{D}$, then

$$\begin{aligned}(Pu)(z) &= \langle Pu, k_z \rangle \\ &= \langle u, k_z \rangle \\ &= \int_{\mathbb{D}} \frac{u(w)}{(1 - z\bar{w})^2} dA(w).\end{aligned}$$

For $f \in L^\infty(\mathbb{D})$, define the *Toeplitz operator* $T_f: L_a^2 \rightarrow L_a^2$ by

$$T_f h = P(fh).$$

Clearly

$$\|T_f\| \leq \|f\|_\infty.$$

If $f \in L^\infty(\mathbb{D})$ and $z \in \mathbb{D}$, then

$$\begin{aligned}\widetilde{T}_f(z) &= \langle T_f K_z, K_z \rangle \\ &= \langle P(fK_z), K_z \rangle \\ &= \langle fK_z, K_z \rangle \\ &= (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - \bar{z}w|^4} dA(w).\end{aligned}$$

Definition: ***Berezin transform of function***

For $f \in L^\infty(\mathbb{D})$, the *Berezin transform* of f is the function $\widetilde{f}: \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$\widetilde{f} = \widetilde{T}_f.$$

Thus if $z \in \mathbb{D}$ then

$$\widetilde{f}(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - \bar{z}w|^4} dA(w).$$

$$\tilde{f}(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - \bar{z}w|^4} dA(w)$$

Because

$$1 = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{1}{|1 - \bar{z}w|^4} dA(w)$$

for all $z \in \mathbb{D}$, think of $\tilde{f}(z)$ as a weighted average of the values of $f(w)$, with most of the weight near z if $|z| \approx 1$.

Thus if $f \in C(\overline{\mathbb{D}})$, then $\tilde{f} \in C(\overline{\mathbb{D}})$ and

$$\tilde{f}|_{\partial\mathbb{D}} = f|_{\partial\mathbb{D}}.$$

Coburn [6]

Suppose $f \in C(\overline{\mathbb{D}})$. Then the following are equivalent:

- 1 T_f is compact.
- 2 $f|_{\partial\mathbb{D}} = 0$.
- 3 $\tilde{f}(z) \rightarrow 0$ as $|z| \uparrow 1$.

Berezin transforms of harmonic functions

$$\tilde{f}(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - \bar{z}w|^4} dA(w)$$

Let H^∞ = bounded analytic functions on \mathbb{D} . If $f \in H^\infty$ and $h \in L^2_a$, then

$$T_f h = fh.$$

If $f \in H^\infty$, then

$$\begin{aligned} \tilde{f}(z) &= \widetilde{T_f}(z) \\ &= (1 - |z|^2)^2 \langle T_f k_z, k_z \rangle \\ &= (1 - |z|^2)^2 \langle f k_z, k_z \rangle \\ &= (1 - |z|^2)^2 f(z) k_z(z) \\ &= f(z). \end{aligned}$$

Thus $f \in H^\infty \implies \tilde{f} = f$.

Thus the real part of each function in H^∞ equals its Berezin transform. This leads to:

If u is a bounded harmonic function on \mathbb{D} , then $\tilde{u} = u$.

Is the converse true?

If $u \in L^\infty(\mathbb{D})$ and $\tilde{u} = u$, is u harmonic? Yes!

Engliš [8]

Suppose $u \in L^\infty(\mathbb{D})$. Then

$$u \text{ is harmonic on } \mathbb{D} \iff \tilde{u} = u.$$

Is the result above valid on the unit ball in \mathbf{C}^n , with harmonic replaced by \mathcal{M} -harmonic?

Ahern, Flores, Rudin [1]: The Engliš result above holds on the unit ball of \mathbf{C}^n if and only if $n \leq 11$.

For $f \in L^\infty(\mathbb{D})$, define the *Toeplitz operator* $T_f: L_a^2 \rightarrow L_a^2$ by

$$T_f h = P(fh),$$

where P is the orthogonal projection of $L^2(\mathbb{D})$ onto L_a^2 .

$$(T_f h)(z) = \int_{\mathbb{D}} \frac{f(w)h(w)}{(1 - z\bar{w})^2} dA(w)$$

for $f \in L^\infty(\mathbb{D})$, $h \in L_a^2$, and $z \in \mathbb{D}$.

$$\begin{aligned} \tilde{f}(z) &= \widetilde{T_f}(z) \\ &= (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - \bar{z}w|^4} dA(w) \end{aligned}$$

for $f \in L^\infty(\mathbb{D})$ and $z \in \mathbb{D}$.

A., Zheng [4]

If S is a finite sum of finite products of Toeplitz operators then the following are equivalent:

- 1 S is compact.
- 2 $\tilde{S}(z) \rightarrow 0$ as $|z| \uparrow 1$.
- 3 $\|SK_z\|_2 \rightarrow 0$ as $|z| \uparrow 1$.

Previously known special cases of $\tilde{f}(z) \rightarrow 0$ as $|z| \uparrow 1 \implies T_f$ is compact:

- f is a nonnegative function on \mathbb{D} . (Zhu [16])
- f is a radial function. (Korenblum and Zhu [9])
- f is uniformly continuous with respect to the hyperbolic metric. (Stroethoff [14])

Hankel operators

Recall that P denotes the orthogonal projection of $L^2(\mathbb{D})$ onto L_a^2 . Thus $I - P$ is the orthogonal projection of $L^2(\mathbb{D})$ onto $L^2(\mathbb{D}) \ominus L_a^2$, which is the orthogonal complement of L_a^2 .

For $g \in L^\infty(\mathbb{D})$, the *Hankel operator*

$$H_g: L_a^2 \rightarrow L^2(\mathbb{D}) \ominus L_a^2$$

is defined by

$$H_g h = (I - P)(gh).$$

Suppose $g \in L^\infty(\mathbb{D})$. Then

$$g \in H^\infty \iff H_g = 0.$$

If $g \in L^\infty(\mathbb{D})$, $h \in L_a^2$, and $z \in \mathbb{D}$, then

$$\begin{aligned}(H_g h)(z) &= g(z)h(z) - (P(gh))(z) \\ &= \int_{\mathbb{D}} \frac{(g(z) - g(w))h(w)}{(1 - z\bar{w})^2} dA(w).\end{aligned}$$

For which $g \in L^\infty(\mathbb{D})$ is H_g compact? Note that H_g does not map L_a^2 into L_a^2 and thus it does not make sense to take the Berezin transform of H_g . However,

H_g is compact $\iff H_g^* H_g: L_a^2 \rightarrow L_a^2$ is compact.

If $f, g \in L^\infty(\mathbb{D})$, then

$$H_f^* H_g = T_{fg} - T_f T_g.$$

Hankel operators with conjugate analytic symbol

A. [2]

Suppose $f \in H^\infty$. Then

$$H_{\bar{f}} \text{ is compact} \iff \lim_{|z| \uparrow 1} (1 - |z|^2) f'(z) = 0.$$

Thus if $f \in H^\infty \cap (\text{Dirichlet space})$, then $H_{\bar{f}}$ is compact.

There exist Blaschke products f with infinitely many zeros such that $\lim_{|z| \uparrow 1} (1 - |z|^2) f'(z) = 0$ (see [12]).

Thus there exist Blaschke products f with infinitely many zeros such that $H_{\bar{f}}$ is compact.

Proof in one direction:

$H_{\bar{f}}$ is compact

$$\iff H_{\bar{f}}^* H_{\bar{f}} \text{ is compact}$$

$$\iff \widetilde{H_{\bar{f}}^* H_{\bar{f}}}(z) \rightarrow 0 \text{ as } |z| \uparrow 1$$

$$\iff \langle H_{\bar{f}}^* H_{\bar{f}} K_z, K_z \rangle \rightarrow 0 \text{ as } |z| \uparrow 1$$

$$\iff \|H_{\bar{f}} K_z\|_2 \rightarrow 0 \text{ as } |z| \uparrow 1$$

$$\iff \|\bar{f} K_z - P(\bar{f} K_z)\|_2 \rightarrow 0 \text{ as } |z| \uparrow 1$$

$$\iff \|\bar{f} K_z - \overline{f(z)} K_z\|_2 \rightarrow 0 \text{ as } |z| \uparrow 1$$

$$\iff \|(f - f(z)) K_z\|_2 \rightarrow 0 \text{ as } |z| \uparrow 1.$$

bound on $(1 - |z|^2)f'(z)$

Suppose $g = \sum_{n=0}^{\infty} a_n z^n \in L_a^2$. Then

$$\|g - g(0)\|_2^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n+1} \geq \frac{|a_1|^2}{2} = \frac{|g'(0)|^2}{2}.$$

Thus

$$(*) \quad |g'(0)| \leq \sqrt{2} \|g - g(0)\|_2.$$

For $z \in \mathbb{D}$, define $\varphi_z: \mathbb{D} \rightarrow \mathbb{D}$ by

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

Then

$$|\varphi_z'(w)| = \frac{1 - |z|^2}{|1 - \bar{z}w|^2}.$$

For $f \in H^\infty$, the operator $H_{\bar{f}}$ is compact

$$\iff \|(f - f(z))K_z\|_2 \rightarrow 0 \text{ as } |z| \uparrow 1.$$

If $z \in \mathbb{D}$ and $f \in H^\infty$ then

$$\begin{aligned} |(f \circ \varphi_z)'(0)| &= |f'(\varphi_z(0))\varphi_z'(0)| \\ &= (1 - |z|^2)|f'(z)|. \end{aligned}$$

Use (*) with $g = f \circ \varphi_z$, getting

$$\begin{aligned} (1 - |z|^2)|f'(z)| &\leq \sqrt{2} \|f \circ \varphi_z - f(z)\|_2 \\ &= \sqrt{2} \|(f - f(z))K_z\|_2. \end{aligned}$$

Thus if $f \in H^\infty$ and $H_{\bar{f}}$ is compact, then

$$\lim_{|z| \uparrow 1} (1 - |z|^2)f'(z) = 0.$$

Bloch space and little Bloch space

Definition: **Bloch space**

The *Bloch space* \mathcal{B} is the set of analytic functions f on \mathbb{D} such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

The *little Bloch space* \mathcal{B}_0 is the set of analytic functions f on \mathbb{D} such that

$$\lim_{|z| \uparrow 1} (1 - |z|^2) f'(z) = 0.$$

The Bloch space \mathcal{B} is a Banach space and \mathcal{B}_0 is a closed subspace with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)|.$$

For $g \in L^2(\mathbb{D})$, define the Hankel operator H_g from H^∞ (with the L_a^2 -norm) to L_a^2 by

$$H_g h = (I - P)(gh).$$

A. [2]

Suppose $f \in L_a^2$. Then

- $H_{\bar{f}}$ is bounded if and only if $f \in \mathcal{B}$.
- $H_{\bar{f}}$ is compact if and only if $f \in \mathcal{B}_0$.
- The Möbius invariant space generated by H^2 is BMOA.
- The Möbius invariant space generated by L_a^2 is \mathcal{B} .

Coifman, Rochberg, and Weiss [7]

$(L_a^1)^* \approx \mathcal{B}$, meaning the dual of L_a^1 can be identified with \mathcal{B} . More precisely, if $f \in \mathcal{B}$ then

$$h \mapsto \lim_{r \uparrow 1} \int_{r\mathbb{D}} h(z) f(\bar{z}) dA(z)$$

defines a bounded linear functional on L_a^1 .

Furthermore, if $\varphi: L_a^1 \rightarrow \mathbf{C}$ is a bounded linear functional, then \exists a unique $f \in \mathcal{B}$ such that

$$\varphi(h) = \lim_{r \uparrow 1} \int_{r\mathbb{D}} h(z) f(\bar{z}) dA(z)$$

for all $h \in L_a^1$.

Also, $(\mathcal{B}_0)^* \approx L_a^1$.

Recall that

$$P: L^2(\mathbb{D}) \rightarrow L_a^2$$

is orthogonal projection.

$$C(\bar{\mathbb{D}}) \subset L^\infty(\mathbb{D}) \subset L^2(\mathbb{D}).$$

$$P(L^\infty(\mathbb{D})) = \mathcal{B}.$$

Also,

$$P(C(\bar{\mathbb{D}})) = \mathcal{B}_0.$$

Littlewood ([10], page 145):

“Bloch's theorem. One of the queerest things in mathematics, and one might judge that only a madman could do it. He was aiming at an elementary proof of Picard's theorem, an impudently damn fool idea. With this as a start it is a just reasonable stroke of insight to conjecture Bloch's theorem. The result once conjectured (and being true), a proof was, of course, bound to emerge sooner or later. But, to keep up the air of farce to the end, the proof itself is crazy.”

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THANK YOU!

