Harmonic Functions from a Complex Analysis Viewpoint

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he accepted an invitation by the Australian National University to take part in a summer talent search modeled on his Ohio State Program and remained active in that January summer program until 1983. Between summer programs in the northern and southern hemispheres, he found time in 1978 and 1979 to help initiate, with Professor P. Roquette, a similar program in Heidelberg.

Arnold Ross has been widely recognized as an outstanding mathematics educator. The long list of his achievements, service and awarded honors include membership of NSF's Advisory Board on Science Education and of CUPM where he chaired a panel on innovations, his being named Outstanding Educator in America (1974–75), receiving Notre Dame's Award of Honor on its 1965 Centennial of Science, Ohio State's Distinguished Teaching and Service Awards, and an Honorary Doctorate from Denison University in 1984.

But the greatest tribute to him is the respect, affection and recognition given him by his former students and by some of our colleagues who were fortunate enough to teach his former students when they reached college or graduate school.

The following small sample of their comments is particularly refreshing for two deep reasons: it gives testimony to the long range effects of an educational experience on people's lives rather than on their end-of-year test scores; and it includes people in many professions, not just mathematicians.

We learned that the fun of doing mathematics—and, by extension, of engaging in other intellectual activities as well—is directly proportional to the intensity of one's commitment.

* * *

Some of us pursue careers in mathematics; others have chosen divergent paths. But there can be few indeed who participated in Dr. Ross's summer programs without being profoundly affected by the experience.

* * *

A. Ross's summer program has no equals in terms of inspiring students with the spirit of mathematical research and individual creativity. He has set an example few of us can meet, but which is of decisive importance to the future and ethos of American Mathematical Education.

HARMONIC FUNCTIONS FROM A COMPLEX ANALYSIS VIEWPOINT

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Recall that a real valued function u defined on an open set in the complex plane is called harmonic if the partial second derivatives of u exist and are continuous and

$$u_{xx} + u_{yy}$$

is identically zero.

Although harmonic functions play a crucial role in several areas of pure and applied mathematics, most mathematicians and mathematics students are more familiar with the elementary properties of analytic functions than of harmonic functions. For example, almost all of us have had to take a test some time in our lives in which we needed to know that an isolated

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singularity of a bounded analytic function is removable. As we will see later, the same theorem holds for harmonic functions, yet this result is not common knowledge among nonspecialists. This paper will study harmonic functions from the comfortable perspective of complex analysis.

We all know that the real and imaginary parts of an analytic function are harmonic. On an open disk the converse is true—every real valued harmonic function is the real part of some analytic function. With this knowledge, the reader should be able to derive most of the *local* properties of harmonic functions from the corresponding properties of analytic functions. For example, every real valued harmonic function is infinitely differentiable, satisfies the mean value property, cannot take on a local maximum or minimum without being constant, etc.

The global behavior of harmonic functions is not so simple to analyze, because on an arbitrary region not every harmonic function is the real part of some analytic function. For example, consider the function $\log|z|$ defined on some open annulus centered at the origin. This function is harmonic, but it is not the real part of any function analytic on the annulus. As we will see later, this is essentially the only example of this phenomenon on an annulus, in the sense that any real valued harmonic function on an annulus is the real part of some function analytic on the annulus plus a real constant times $\log|z|$.

The main theme of this paper is that on finitely connected regions every real valued harmonic function is the real part of some function analytic on the region plus some logarithm terms; the Logarithmic Conjugation Theorem in the next section will give a precise statement. For our purposes, it is best to define a finitely connected region Ω to be a nonempty, connected open subset of the complex plane whose complement (with respect to the complex plane) has only finitely many bounded connected components. For obvious geometrical reasons (see Fig. 1), each bounded component of the complement of Ω is called a hole of Ω. A finitely connected region that has no holes is of course called simply connected.

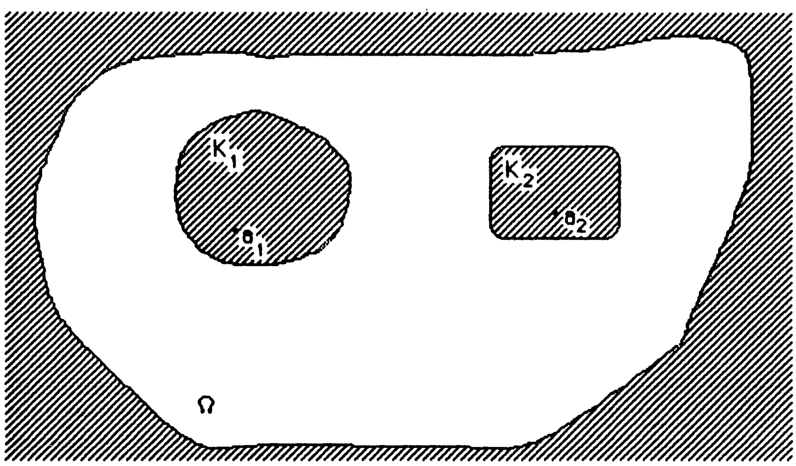

FIG. 1

Harmonic Conjugates. If Ω is simply connected and u is a real valued harmonic function on Ω, then there is an analytic function f on Ω such that $u = \operatorname{Re} f$; here Re denotes the real part, and Im will denote the imaginary part. The function $\operatorname{Im} f$ is called the harmonic conjugate of u, and except for the addition of a constant, it is unique.

Harmonic conjugates are so useful that many mathematicians use them even when they don't exist (on regions that are not simply connected). Thus it is possible to read about a harmonic conjugate that is a "multi-valued function with periods." I have always been mystified by this terminology. Any good high school student knows that if g is a function and z is an element of the domain of g, then $g(z)$ is a single object. Thus, by definition, a function cannot be multi-valued. Complex analysts will recognize that Riemann surfaces provide the rigorous mathematics lurking behind the "multi-valued functions" that appear. However, "multi-valued functions" are often discussed in texts where the reader is unfamiliar with the concept of a Riemann surface, and I suspect that many people (including me) are confused by the nebulous concept of a "multi-valued function." The approach I will present here allows us to obtain the results we want without resorting to either "multi-valued functions" or Riemann surfaces.

The following theorem, which is the main tool of this paper, can replace the "multi-valued function" approach to harmonic conjugation and simplify many proofs. Since this theorem states that each function on a finitely connected region has a harmonic conjugate, provided we first subtract some logarithmic terms, I call it the Logarithmic Conjugation Theorem.

LOGARITHMIC CONJUGATION THEOREM. *Suppose* Ω *is a finitely connected region, with* $K_1, \dots, K_N$ *denoting the bounded components of the complement of* Ω. *For each* j, *let* a_j *be a point in* K_j. *If* u *is a real valued harmonic function on* Ω, *then there exist an analytic function* f *on* Ω *and real numbers* $c_1, \dots, c_N$ *such that*

$$u(z) = \operatorname{Re} f(z) + c_1 \log|z - a_1| + \cdots + c_N \log|z - a_N|$$

for every z *in* Ω.

With this theorem, there is no need to talk about "the periods of the multi-valued harmonic conjugate of u," but if one were to do so, the periods would turn out to be exactly the numbers $c_1, \dots, c_N$.

The only place I have been able to find the Logarithmic Conjugation Theorem written down with a proof is in an old paper of Walsh [**18**], pages 518 and 527. Walsh's proof uses a special version of Green's formula that is valid for harmonic functions. The elementary proof I will present here uses only the Cauchy Integral Theorem for analytic functions, so it should be more accessible to modern audiences. After proving the Logarithmic Conjugation Theorem, we will see how it can be used to study isolated singularities of harmonic functions, help solve the Dirichlet problem on annuli, and lead to a short proof of the conformal mapping theorem for doubly connected regions.

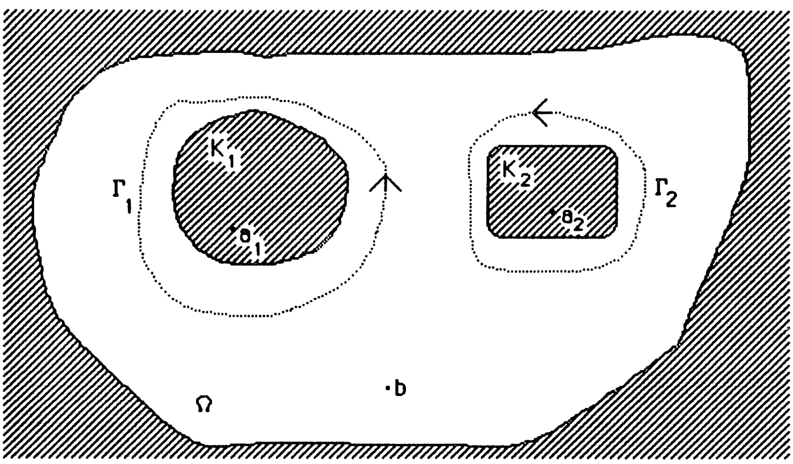

FIG. 2

Proof of the Logarithmic Conjugation Theorem. Define a function h on Ω by

$$h(z) = u_x(z) - iu_y(z).$$

The function h is analytic on Ω; this is proved by verifying the Cauchy-Riemann equations, which in the case of h are

$$u_{xx} = -u_{yy},$$
$$u_{xy} = -(-u_{yx}).$$

Of course, the first equation holds because u is harmonic, and the second equation holds because the order of partial differentiation doesn't matter.

For each j $(1 \leqslant j \leqslant N)$, let Γ_j be a curve in Ω that surrounds the hole K_j; see Fig. 2. (The technical definition of "surrounds" is that Γ_j has winding number one about each point of K_j and winding number zero about all the other holes of Ω.) Now define c_j by

$$c_j = (1/2\pi i)\int_{\Gamma_j} h(w)\, dw.$$

To see that each c_j is a real number, note that

$$\begin{aligned}\operatorname{Im} c_j &= (-1/2\pi)\operatorname{Re}\int_{\Gamma_j} h(z)\, dz \\ &= (-1/2\pi)\operatorname{Re}\int_{\Gamma_j}(u_x - iu_y)(dx + idy) \\ &= (-1/2\pi)\int_{\Gamma_j} u_x\, dx + u_y\, dy.\end{aligned}$$

The last integral is an exact differential over a closed curve, so it equals zero, and thus c_j is real.

Now fix a point b in Ω, and define a function f on Ω by

$$f(z) = \int_b^z h(w) - \frac{c_1}{w - a_1} - \cdots - \frac{c_N}{w - a_N}\, dw,$$

where the integration is taken over any path in Ω from b to z. To show that f is well defined, we must check that the integral above is independent of the path from b to z. But given two different paths from b to z, we can reverse the direction of the second path, getting a closed curve in Ω; see Fig. 3.

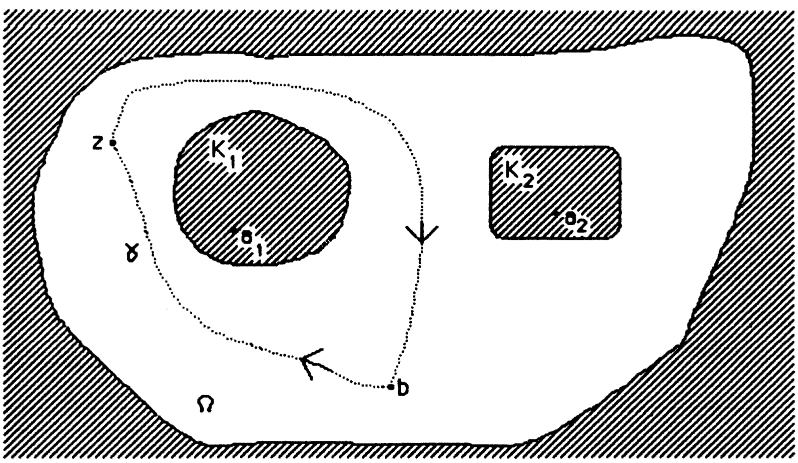

FIG. 3

Thus we need only show that

$$(*) \qquad 0 = \frac{1}{2\pi i}\int_{\gamma} h(w) - \frac{c_1}{w - a_1} - \cdots - \frac{c_N}{w - a_N}\, dw$$

for each closed curve γ in Ω. However, if m_j denotes the winding number of γ about K_j, then the Cauchy Integral Theorem (along with the definition of c_j) says that

$$(1/2\pi i)\int_\gamma h(w)\,dw = m_1 c_1 + \cdots + m_N c_N,$$

while the definition of winding number implies that

$$\frac{1}{2\pi i}\int_\gamma \frac{c_1}{w-a_1} + \cdots + \frac{c_N}{w-a_N}\,dw = m_1 c_1 + \cdots + m_N c_N.$$

The last two equations show that $(*)$ holds.

Now that we know f is well defined, it is clear (just examine the difference quotient) that f is analytic on Ω and

$$f'(z) = h(z) - c_1/(z-a_1) - \cdots - c_N/(z-a_N).$$

Having assembled our analytic function f and real numbers $c_1, \dots, c_N$, we are ready to verify that the conclusion of the Logarithmic Conjugation Theorem holds. For convenience, define a function q on Ω by

$$q(z) = \operatorname{Re} f(z) + c_1 \log|z-a_1| + \cdots + c_N \log|z-a_N|.$$

Add a constant to f so that u and q agree at some point of Ω, say at b. We will finish the proof by showing that $u_x(z) = q_x(z)$ and $u_y(z) = q_y(z)$ for all points z in Ω, so that $u = q$.

Differentiation is a local operation, and since locally $\log(z-a_j)$ is analytic on Ω, we have

$$\begin{aligned} q_x(z) &= \left[\operatorname{Re} f(z) + c_1 \log|z-a_1| + \cdots + c_N \log|z-a_N|\right]_x \\ &= \left[\operatorname{Re}\{f(z) + c_1 \log(z-a_1) + \cdots + c_N \log(z-a_N)\}\right]_x \\ &= \operatorname{Re}\left[\{f(z) + c_1 \log(z-a_1) + \cdots + c_N \log(z-a_N)\}_x\right]. \end{aligned}$$

For analytic functions, the partial derivative with respect to x is the same as the complex derivative, so the last equation becomes

$$q_x(z) = \operatorname{Re}\left[f'(z) + c_1/(z-a_1) + \cdots + c_n/(z-a_n)\right].$$

Plugging the formula we found earlier for f' into the above equation gives

$$q_x(z) = \operatorname{Re} h(z),$$

and recalling the definition of h, we get

$$q_x(z) = u_x(z).$$

Finally, computing q_y is done in the same fashion, except now we note that taking the y partial derivative of an analytic function gives i times the complex derivative. So we get

$$q_y(z) = \operatorname{Re}(ih(z)),$$

and again referring to the definition of h gives

$$q_y(z) = u_y(z).$$

Thus we have completed the proof of the Logarithmic Conjugation Theorem.

Isolated Singularities. To study the behavior of harmonic functions near isolated singularities, we need only consider harmonic functions defined on punctured disks. So let $\mathbb{D}$ denote the open unit disk in the complex plane, and let $\mathbb{D}'$ denote the open unit disk with the origin removed. The complement of $\mathbb{D}'$ has only one bounded component. Thus the Logarithmic Conjugation Theorem (with $a_1 = 0$) tells us that any real valued harmonic function u on $\mathbb{D}'$ can be written in the form

$$u(z) = \operatorname{Re} f(z) + c \log|z|,$$

where f is analytic on $\mathbb{D}'$ and c is a real number. Using the classification of isolated singularities of analytic functions into removable singularities, poles, and essential singularities, we can now see the possible behaviors of harmonic functions near isolated singularities.

In particular, suppose our harmonic function u is bounded on $\mathbb{D}'$. Rewrite the above equation as

$$\operatorname{Re} f(z) = u(z) - c \log|z|.$$

Suppose first that c is positive. Since u is bounded, this means that $\operatorname{Re} f(z) \to \infty$ as $z \to 0$. This behavior is clearly impossible if f has a removable singularity at 0. Also, f cannot have a pole at 0, because then $f(\mathbb{D}')$ would include the complement of some disk, and in particular $f(\mathbb{D}')$ would include a sequence whose real part tends to $-\infty$. Finally, f cannot have an essential singularity at 0, because then there would be a sequence tending to 0 on which f (and hence $\operatorname{Re} f$) stays bounded. Since no other possibilities remain, we conclude that c cannot be positive. Similarly, c cannot be negative. Thus $c = 0$ and so $u = \operatorname{Re} f$. This means that $\operatorname{Re} f$ is bounded. Again this is impossible if f has either a pole or an essential singularity at 0, and so f has a removable singularity at 0. Of course this means that u also has a removable singularity at 0. We have thus proved the following theorem.

Isolated Singularity, Bounded Function. *If a harmonic function is bounded near an isolated singularity, then that singularity is removable.*

The above theorem was first published by Schwarz [**16**], p. 252, whose proof used the series representation for harmonic functions on annuli (discussed in the next section of this paper). Many years later Picard [**13**], who seemed not to know that Schwarz had already published a proof, gave a proof using a "multi-valued harmonic conjugate with a period," precisely the object that the Logarithmic Conjugation Theorem allows us to avoid. Perhaps Lebesgue disliked "multi-valued functions"; three weeks after Picard presented his proof to the Académie des Sciences in Paris, Lebesgue (also unaware of Schwarz's proof) presented another proof [**11**] to the same body. Lebesgue's proof requires knowledge of the solvability of the Dirichlet problem for an annulus (discussed in the next section of this paper).

Now let's see how the Logarithmic Conjugation Theorem can help in the study of some important spaces of analytic functions. For an arbitrary region Ω and a positive number p, the much studied Hardy space $H^p(\Omega)$ is defined to be the set of analytic functions h on Ω such that there exists a harmonic function u on Ω with

$$|h(z)|^p \leqslant u(z)$$

for all z in Ω. If Ω is the unit disk $\mathbb{D}$, this definition is equivalent to the usual definition involving integrals around concentric circles. A good reference on Hardy spaces is Fisher's recent book [**6**].

The Logarithmic Conjugation Theorem can be especially useful when investigating properties of Hardy spaces on finitely connected regions. Here is an example of an important theorem from that subject: Let Ω be a bounded finitely connected region with smooth boundary. Then each continuous real valued function on $\partial\Omega$ can be uniformly approximated by functions of the form

$$\operatorname{Re} f(z) + c_1 \log|z - a_1| + \cdots + c_N \log|z - a_N|,$$

where f is a rational function whose poles are all in the exterior of Ω, where each c_j is a real number, and where each a_j is in the interior of the corresponding hole of Ω (see Fig. 1).

A proof of the result above (see for example [**6**], Theorem 2.1) would take us beyond the scope of this paper, but the reader who has noted that the theorem above is formally similar to the Logarithmic Conjugation Theorem will not be surprised that the Logarithmic Conjugation Theorem is useful in simplifying the standard proof.

A function in a Hardy space need not be bounded, so the following theorem is stronger than the usual result about isolated singularities of bounded analytic functions. As we will see in the proof, the Logarithmic Conjugation Theorem is an excellent tool for dealing with isolated singularities of Hardy space functions.

ISOLATED SINGULARITY; HARDY SPACE FUNCTION. *An isolated singularity of a function that belongs to a Hardy space is removable.*

To prove this theorem, we can fix a positive number p and assume that h is a function in $H^p(\mathbb{D}')$; we need to show that h is analytic at 0. By definition, there is a function u harmonic on $\mathbb{D}'$ such that

$$|h(z)|^p \leqslant u(z)$$

for each z in $\mathbb{D}'$. As usual, use the Logarithmic Conjugation Theorem to write u in the form

$$u(z) = \operatorname{Re} f(z) + c \log|z|,$$

where f is analytic on $\mathbb{D}'$ and c is a real number. Now

$$|e^{-f(z)}| = e^{-\operatorname{Re} f(z)} = e^{-u(z) + c\log|z|} = |z|^c e^{-u(z)},$$

so

$$|e^{-f(z)}|\,|z|^{-c} = e^{-u(z)} \leqslant 1.$$

If f had either a pole or an essential singularity at 0, then there would be a sequence tending to 0 on which e^{-f} is bounded and also a sequence tending to 0 on which e^{-f} is unbounded; this behavior would mean that e^{-f} has an essential singularity at 0. However, the last line of the previous paragraph shows that e^{-f} has either a pole or a removable singularity at 0. Thus we can conclude that f has a removable singularity at 0.

So now we know that

$$|h(z)|^p \leqslant \operatorname{Re} f(z) + c \log|z|,$$

where f is analytic on $\mathbb{D}$. Since f is bounded near 0, the above inequality implies that $|z|^p|h(z)|^p \to 0$ as $z \to 0$. Thus $zh(z) \to 0$ as $z \to 0$, which implies that h is analytic at 0, as desired.

Analysis on Annuli. From now on, $\mathscr{A}$ will denote an open annulus centered at the origin. In this section, we will use the Logarithmic Conjugation Theorem to investigate the integrals of a harmonic function around concentric circles and to give a series representation for the most general harmonic function on $\mathscr{A}$. Then we will use this information to guess the solution to the Dirichlet problem for annuli. Until we begin talking about the Dirichlet problem, everything that is said about annuli will also apply to punctured disks.

If u is a real valued harmonic function on $\mathscr{A}$, then the Logarithmic Conjugation Theorem tells us that there is a function f analytic on $\mathscr{A}$ and a real number c such that

$$u(z) = \operatorname{Re} f(z) + c \log|z|$$

for every z in $\mathscr{A}$.

Suppose we fix a positive number r in $\mathscr{A}$. Then from the equation above we get

$$\begin{aligned}(1/2\pi)\int_0^{2\pi} u(re^{i\theta})\,d\theta &= c\log r + \operatorname{Re}\left[(1/2\pi)\int_0^{2\pi} f(re^{i\theta})\,d\theta\right]\\ &= c\log r + \operatorname{Re}\left[(1/2\pi i)\int_{\partial(r\mathbb{D})} f(z)/z\,dz\right],\end{aligned}$$

where as usual $\mathbb{D}$ denotes the unit disk. The Cauchy Integral Theorem tells us that the last term in the equation above is independent of r (in fact, the last term equals the real part of the constant term in the Laurent series expansion of f). Thus we have the following result:

INTEGRALS AROUND CIRCLES. *If u is harmonic on $\mathscr{A}$, then*

$$\int_0^{2\pi} u(re^{i\theta})\,d\theta$$

is a linear function of $\log r$.

The fact above has been noted elsewhere (see [**1**], Theorem 20, p. 265 and [**15**], p. 21), but the proof given here using the Logarithmic Conjugation Theorem seems considerably easier than other proofs.

Now let's use the Logarithmic Conjugation Theorem to find a series representation for the most general real valued harmonic function u on $\mathscr{A}$. As usual, first we write

$$u(z) = \operatorname{Re} f(z) + c \log|z|.$$

Since f is analytic on the annulus $\mathscr{A}$, we can express f as a Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} 2a_n z^n,$$

where for convenience the coefficient of z^n is called $2a_n$. Now

$$u(z) = \left[f(z) + \overline{f(z)}\right]/2 + c\log|z|.$$

Let $z = re^{i\theta}$, replace $f(z)$ with its Laurent series, and evaluate $\overline{f(z)}$ by taking the complex conjugate of the Laurent series for $f(z)$, and then replace n with $-n$ in the summation, getting the following formula:

SERIES REPRESENTATION. *If u is a real valued harmonic function on an annulus $\mathscr{A}$, then u has the form*

$$(**) \qquad u(re^{i\theta}) = c\log r + \sum_{n=-\infty}^{\infty}\left(a_n r^n + \overline{a_{-n}}\,r^{-n}\right)e^{in\theta}.$$

The infinite sum above converges absolutely for each $re^{i\theta}$ in $\mathscr{A}$ and uniformly on compact subsets of $\mathscr{A}$ (because the Laurent series for f has these properties). Other derivations of this useful formula are given by Saks and Zygmund [**14**], Chapter 10, Section 3, and Heins [**8**], pages 56–57.

The above formula can help us see that the Dirichlet problem is solvable for annuli. Recall that for a bounded region Ω, we say the Dirichlet problem is solvable on Ω if for each continuous real valued function U on $\partial\Omega$, there is a continuous real valued function u on the closure of Ω such that u is harmonic on Ω and $u|\partial\Omega = U$.

For the unit disk $\mathbb{D}$, the Dirichlet problem is explicitly solved by the Poisson integral expressing u in terms of U:

$$u(re^{i\theta}) = \frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{U(e^{i(\theta-t)})}{1-2r\cos t + r^2}\,dt.$$

The example of the disk leads us to suspect that for an annulus $\mathscr{A}$, we could find u from U by integrating U on the two circles of $\partial\mathscr{A}$ against an appropriate kernel. This is correct, but the kernel for $\mathscr{A}$ is not an elementary function (like $(1-r^2)/[1-2r\cos t + r^2]$ for the disk), but a more complicated expression involving elliptic functions; see [**17**] and [**10**]. Alternatively, the annulus can be lifted to its universal covering space (represented as an infinite strip) and then we can integrate on lines rather than circles. This procedure, worked out by Sarason [**15**], pages 21–23, and more recently by Wang [**19**], Theorem 1, gives a kernel on the boundary of the strip that is an elementary (but complicated) function.

We will take a different approach here, using the Series Representation to lead us to an elementary proof that the Dirichlet problem is solvable for an annulus. To begin, let s denote the inner radius of $\mathscr{A}$ and t denote the outer radius of $\mathscr{A}$. Consider a real valued function U on $\partial\mathscr{A}$ of the form

$$U(se^{i\theta}) = \sum_{n=-N}^{N} b_n e^{in\theta}$$

$$U(te^{i\theta}) = \sum_{n=-N}^{N} d_n e^{in\theta}$$

Since U is real valued, b_{-n} must equal the complex conjugate of b_n (just equate U to $\overline{U}$), and similarly for the coefficients d_n. We will first show that the Dirichlet problem is solvable for functions U of the above form, and then show why this implies that the Dirichlet problem is solvable for every continuous real valued function on $\partial\mathscr{A}$.

Compare our formula (* *) for the most general harmonic function on $\mathscr{A}$ with the expressions defining U; it is clear that for n positive we want to choose a_n and a_{-n} so that

$$a_n s^n + \overline{a_{-n}} s^{-n} = b_n,$$

$$a_n t^n + \overline{a_{-n}} t^{-n} = d_n.$$

For each positive n, we can consider these equations as two linear equations (with a_n and $\overline{a_{-n}}$ being the unknown quantities). Since $s \neq t$, the Jacobian of this system is nonzero, and so we can solve for a_n and $\overline{a_{-n}}$.

The relationship between b_n and b_{-n} (and d_n and d_{-n}) insures that we would get the same values for a_n and a_{-n} if we considered these equations for n negative. What happens when n equals zero? Then the Jacobian of the system is 0, and there is not necessarily a solution to the equations. However, we are rescued by the logarithm term in the formula (* *) for a harmonic function on $\mathscr{A}$. Now the equations we need to solve are

$$c \log s + 2a_0 = b_0,$$

$$c \log t + 2a_0 = d_0.$$

These two equations (with c and a_0 being the unknown quantities) have a nonzero Jacobian, and so we can solve for c and a_0.

At this point we know $\{a_n\}$ and c; the function u that solves the Dirichlet problem for U is just

$$u(re^{i\theta}) = c \log r + \sum_{n=-N}^{N} \left(a_n r^n + \overline{a_{-n}} r^{-n}\right) e^{in\theta}.$$

The interested reader can simply solve the equations to write $\{a_n\}$ and c explicitly in terms of $\{b_n\}$ and $\{d_n\}$.

Now that we know the Dirichlet problem is solvable for functions of the form considered above, we can easily deal with an arbitrary continuous real valued function U on $\partial\mathscr{A}$. By the Stone-Weierstrass Theorem, there is a sequence $\{U_n\}$ of functions on $\partial\mathscr{A}$ of the above form such that U_n converges uniformly to U on $\partial\mathscr{A}$. Let u_n denote the solution to the Dirichlet problem for U_n. The maximum modulus theorem for harmonic functions implies that $\{u_n\}$ is a uniform Cauchy sequence on the closure of $\mathscr{A}$, and it is clear that the limit of this sequence solves the Dirichlet problem for U, and so we are done.

Heins [**9**], pages 301–302, also uses the Stone-Weierstrass Theorem to pass from trigonometric polynomials to arbitrary functions in solving the Dirichlet problem. The advantage of the approach presented here is that the Series Representation allows us to guess naturally the solution for the trigonometric polynomials.

Rather than show that the Dirichlet problem is solvable on $\mathscr{A}$ by dealing with a dense set of functions, an alternative would be to solve directly the problem for an arbitrary continuous function on $\partial\mathscr{A}$, although the verification requires some knowledge about Fourier series. The idea, still motivated by the Series Representation, is to take an arbitrary continuous function U on $\partial\mathscr{A}$, and write it in the form

$$U(se^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta},$$

$$U(te^{i\theta}) = \sum_{n=-\infty}^{\infty} d_n e^{in\theta}.$$

Here the infinite sum converges in the L^2 norm (with respect to arc length measure on $\partial\mathscr{A}$). Now simply solve the same equations as with the previous method to find $\{a_n\}$ and c. The harmonic function u on $\mathscr{A}$ that solves the Dirichlet problem for U is then given by

$$u(re^{i\theta}) = c\log r + \sum_{n=-\infty}^{\infty} \left(a_n r^n + \overline{a_{-n}}\, r^{-n}\right) e^{in\theta}.$$

Conformal Mapping. A region Ω is called doubly connected if its complement (with respect to the complex plane) has precisely one bounded connected component. Of course every annulus is doubly connected. Two annuli are conformally equivalent if and only if their outer radius/inner radius ratios are equal (see [**7**], Chapter 5, Section 1, Theorem 2).

In this section we will use the Logarithmic Conjugation Theorem to give an easy proof that any doubly connected region is conformally equivalent to some annulus. So that we can have a clean statement of the correct theorem, for this section an annulus is defined to be a set of the form $\{z \in \mathbb{C}\colon s < |z| < t\}$, where s is a non-negative number (possibly 0) and t is a positive number or ∞.

DOUBLY CONNECTED MAPPING THEOREM. *Let Ω be a doubly connected region. Then there is an annulus $\mathscr{A}$ and a one-to-one analytic function g on Ω that maps Ω onto $\mathscr{A}$.*

The proof of the Doubly Connected Mapping Theorem given by Courant [**5**], Chapter 1, Section 7, is somewhat similar in spirit to the proof presented here, except that Courant uses a "multi-valued harmonic conjugate with a period," which the Logarithmic Conjugation Theorem will allow us to avoid. Bieberbach [**2**], Chapter 5, Section 26, gives a proof using Riemann surfaces (and leaves out the hard details). Other proofs of the Doubly Connected Mapping Theorem can be found in the books of Nehari [**12**], Chapter 7, and Goluzin [**7**], Chapter 5, Section 1.

In the last section we showed that the Dirichlet problem is solvable on nondegenerate annuli. For our proof of the Doubly Connected Mapping Theorem we will need to know that the Dirichlet problem is solvable for any bounded doubly connected region with smooth boundary. This is really not so difficult to prove. Proofs that fit within the context of a standard complex variables course are given in the texts of Ahlfors [**1**], Chapter 6, Theorem 9, and Conway [**4**], Chapter 10, Corollary 4.17. Browder [**3**], Theorem 3.4.10, gives a proof that is easier, although it requires more machinery.

Proof of the Doubly Connected Mapping Theorem. As usual, we can assume that our doubly connected region Ω is bounded and has a smooth boundary. The standard easy proof that any doubly connected region, except the punctured disk or the punctured plane, is conformally equivalent to a bounded region with smooth boundary can be found in [**1**], pages 252–253. Actually, our proof does not really require a smooth boundary; however, this assumption does make it easier to prove that the Dirichlet problem is solvable for Ω.

Let K denote the bounded component of $\mathbb{C} \sim \Omega$. Without loss of generality we can assume that $0 \in K$. For obvious reasons, we refer to ∂K as the inner boundary of Ω and $\partial(\Omega \cup K)$ as the outer boundary of Ω; see Fig. 4. Let u be the continuous real valued function on the closure

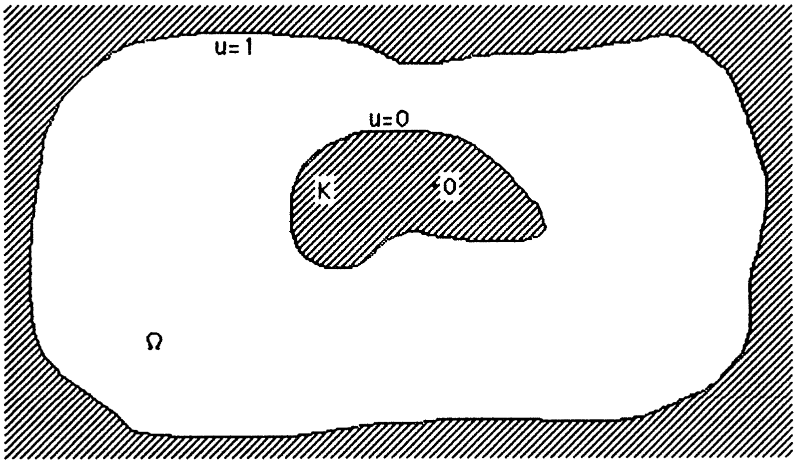

FIG. 4

of Ω that equals 0 on the inner boundary of Ω and equals 1 on the outer boundary of Ω and is harmonic on Ω. Use the Logarithmic Conjugation Theorem to write u in the form

$$u(z) = c \log|z| + \operatorname{Re} f(z),$$

where c is a real number and f is analytic on Ω.

Define a function g on Ω by

$$g(z) = ze^{f(z)/c}.$$

Of course g is analytic on Ω (because f is analytic), and we will see that g maps Ω onto an annulus in a one-to-one fashion. (Let's assume that c is positive, so in particular we don't have to worry about dividing by zero. We will see later how to take care quickly of the cases where c is zero or negative.)

Note that

$$\log|g(z)| = \log|z| + (\operatorname{Re} f(z)/c) = u(z)/c,$$

and since u is between 0 and 1, we can conclude that $|g(z)|$ is between 1 and $e^{1/c}$. In other words, g maps Ω into the annulus $\mathscr{A}$ defined by

$$\mathscr{A} = \{ w \in \mathbf{C} : 1 < |w| < e^{1/c} \}.$$

To see that g covers each point of $\mathscr{A}$ precisely once, fix $w \in \mathscr{A}$. Let γ_0 and γ_1 be closed curves in Ω as in Fig. 5. More precisely, γ_0 should have winding number -1 about each point of K, and γ_0 should be close enough to the inner boundary of Ω so that each point of $g(\gamma_0)$ is smaller in absolute value than $|w|$ (see Fig. 6); this last condition can be satisfied because u is continuous on the closure of Ω and u equals 0 on the inner boundary of Ω (so $|g(z)|$ is near 1 for z near ∂K). The curve γ_1 should be chosen to have winding number one about each point of $K \cup \gamma_0$, and γ_1 should be close enough to the outer boundary of Ω so that each point of $g(\gamma_1)$ is larger in absolute value than $|w|$ (see Figs. 5 and 6); this last condition can be satisfied because u equals 1 on the outer boundary of Ω.

It is clear from Fig. 5 what we mean by the region between γ_0 and γ_1; technically, this region is defined to be the set of points about which $\gamma_0 \cup \gamma_1$ has winding number one. The argument principle states that the number of times g takes on the value w in the region between γ_0 and γ_1 equals the winding number of $g(\gamma_0) \cup g(\gamma_1)$ about w. By our choice of γ_0, the curve $g(\gamma_0)$ lies in a disk disjoint from w, and so $g(\gamma_0)$ winds zero times around w; see Fig. 6. Our construction of γ_1 shows that the disk of radius $|w|$ centered at 0 is disjoint from $g(\gamma_1)$, so the winding number of

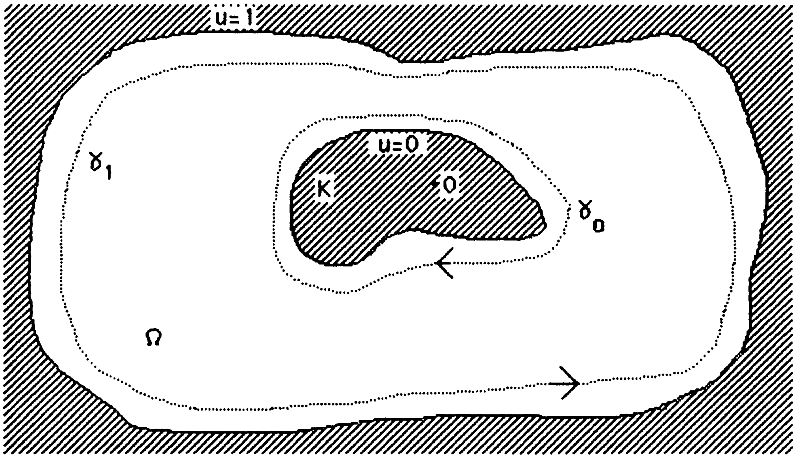

FIG. 5

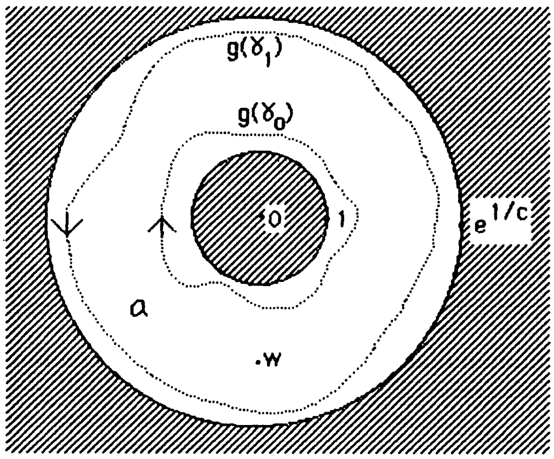

FIG. 6

$g(\gamma_1)$ about w is the same as the winding number of $g(\gamma_1)$ about 0; see Fig. 6. As usual, the winding number of $g(\gamma_1)$ about 0 is equal to the change in $(1/2\pi)\log g(z)$ as z goes once around γ_1. The definition of g shows that

$$(1/2\pi)\log g(z) = (1/2\pi)f/c + (1/2\pi)\log z.$$

Since f is analytic in Ω, the change in $(1/2\pi)f/c$ around γ_1 is zero. The change in $(1/2\pi)\log z$ around γ_1 is of course just the winding number of γ_1 around 0, and recalling that $0 \in K$ and that γ_1 winds once around each point of K, we conclude that this number equals one.

Thus g takes on the value w precisely once in the region between γ_0 and γ_1. Since we can choose γ_0 and γ_1 to be arbitrarily close to the inner and outer boundaries of Ω, we conclude that g takes on the value w precisely once on Ω. In other words, g is a one-to-one map of Ω onto the annulus $\mathcal{A}$.

Thus the proof is completed except for one minor detail that we postponed; we assumed that the constant c was positive. (The way we have defined the function u in this proof always forces c to be positive, but it is easier to deal with the other cases than to prove this.) If c is negative, simply interchange the roles of $e^{1/c}$ and 1 in defining the annulus $\mathcal{A}$ (so $e^{1/c}$ is now the inner radius rather than the outer radius), and the above proof works fine. If c equals zero, then change the definition of g given in the proof to $g(z) = e^{f(z)}$; the same proof now shows that g maps Ω in a one-to-one fashion onto the annulus $\{w \in \mathbb{C}: 1 < |w| < e\}$. Now the proof is finished.

The applications of the Logarithmic Conjugation Theorem are far from exhausted, and I hope that those of you who attempt to find additional uses for this theorem will have as much fun with it as I have had.

References

1. Lars V. Ahlfors, Complex Analysis, 3rd ed., McGraw-Hill, New York, 1979.
2. L. Bieberbach, Conformal Mapping, Chelsea, New York, 1953.
3. Andrew Browder, Introduction to Function Algebras, Benjamin, New York, 1969.
4. John B. Conway, Functions of One Complex Variable, 2nd ed., Springer-Verlag, New York, 1978.
5. R. Courant, Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces, Interscience, New York, 1950.
6. Stephen D. Fisher, Function Theory on Planar Domains, Wiley Interscience, New York, 1983.
7. G. M. Goluzin, Geometric Theory of Functions of a Complex Variable, Translations of Mathematical Monographs, vol. 26, Amer. Math. Soc., Providence, 1969.
8. Maurice Heins, Selected Topics in the Classical Theory of Functions of a Complex Variable, Holt, Rinehart and Winston, New York, 1962.
9. ______, Complex Function Theory, Academic Press, New York, 1968.
10. Yûsaku Komatu, Sur la représentation de Villat pour les fonctions analytiques définies dans un anneau circulaire concentrique, Proc. Japan Academy, 21 (1945) 94–96.
11. Henri Lebesgue, Sur les singularités des fonctions harmoniques, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, 176 (1923) 1097–1099.
12. Zeev Nehari, Conformal Mapping, McGraw-Hill, New York, 1952.
13. Emile Picard, Deux théorèmes élémentaires sur les singularités des fonctions harmoniques, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, 1976 (1923) 933–935.
14. S. Saks and A. Zygmund, Analytic Functions, 3rd ed., Elsevier Publishing, Amsterdam, and Polish Scientific Publishers, Warsaw, 1971.
15. Donald Sarason, The H^p spaces of an annulus, Memoirs of the Amer. Math. Soc., 56 (1965).
16. H. A. Schwarz, Zur Integration der partiellen Differential-gleichung $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, J. Reine Angew. Math., 74 (1872) 218–253.
17. H. Villat, Le problème de Dirichlet dans aire annulaire, Rend. Circ. Mat. Palermo, 33 (1912) 134–175.
18. J. L. Walsh, The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, Bull. Amer. Math. Soc., 35 (1929) 499–544.
19. Hwai-chiuan Wang, Real Hardy spaces of an annulus, Bull. Australian Math. Soc., 27 (1983) 91–105.

164.

MISCELLANEA

Mathematical Life?

I do not remember ever having seen a sustained argument by any author which, starting from philosophical or theological premises likely to meet with general acceptance, reached the conclusion that a praiseworthy ordering of one's life is to devote it to research in mathematics.

—Sir Edmund Whittaker (1873–1956). The quotation is from *Scientific American*, Volume 183 (September 1950), page 42.