

Symbolic Manipulation of Harmonic Functions

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If u is defined on an open subset of \mathbf{R}^n , then

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

u is called *harmonic* if

$$\Delta u \equiv 0.$$

Example: $\|x\|^{2-n}$ is harmonic on $\mathbf{R}^n \setminus 0$.

Example: If $\zeta \in \mathbf{R}^n$ and $\|\zeta\| = 1$ then

$$\frac{1 - \|x\|^2}{\|x - \zeta\|^n}$$

is harmonic on $\mathbf{R}^n \setminus \zeta$.



Dirichlet Problem: Suppose Ω is an open subset of \mathbf{R}^n . Given $f \in C(\partial\Omega)$, find $u \in C(\overline{\Omega})$ such that

$$u \text{ is harmonic on } \Omega \quad \text{and} \quad u|_{\partial\Omega} = f.$$



Johann Dirichlet (1805-1859)

Let B be the open unit ball in \mathbf{R}^n .

Let σ be surface area measure on the unit sphere ∂B , normalized so that $\sigma(\partial B) = 1$.

Theorem: Suppose $f \in C(\partial B)$. Define u on \bar{B} by

$$u(x) = \begin{cases} \int_{\partial B} \frac{1 - \|x\|^2}{\|x - \zeta\|^n} f(\zeta) d\sigma(\zeta) & \text{if } x \in B \\ f(x) & \text{if } x \in \partial B. \end{cases}$$

Then u is harmonic on B and $u \in C(\bar{B})$.

Surprising result: If f is a polynomial, then so is u .



Siméon Poisson (1781-1840)

\mathcal{P}_m is the set of polynomials on \mathbf{R}^n with degree at most m . Fix $b = (b_1, \dots, b_n) \in \mathbf{R}^n$, with each $b_j > 0$. For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, let

$$\|bx\|^2 = b_1^2 x_1^2 + \dots + b_n^2 x_n^2.$$

Let E be the ellipsoid defined by

$$E = \{x \in \mathbf{R}^n : \|bx\|^2 < 1\}.$$

Theorem: If $f \in \mathcal{P}_m$, then there exists $g \in \mathcal{P}_{m-2}$ such that

$$f + (1 - \|bx\|^2)g$$

is harmonic.

Note that

$$(f + (1 - \|bx\|^2)g)|_{\partial E} = f|_{\partial E}.$$

Thus $f + (1 - \|bx\|^2)g$ solves the Dirichlet problem on E with boundary function $f|_{\partial E}$.

Theorem: If $f \in \mathcal{P}_m$, then there exists $g \in \mathcal{P}_{m-2}$ such that

$$f + (1 - \|bx\|^2)g$$

is harmonic.

Proof: We need to show that there exists $g \in \mathcal{P}_{m-2}$ such that

$$\Delta((1 - \|bx\|^2)g) = -\Delta f.$$

Define a linear map $L : \mathcal{P}_{m-2} \rightarrow \mathcal{P}_{m-2}$ by

$$Lg = \Delta((1 - \|bx\|^2)g).$$

If $Lg = 0$, then $(1 - \|bx\|^2)g$ is a harmonic function on E that equals 0 on ∂E . Thus $(1 - \|bx\|^2)g = 0$ and hence $g = 0$.

Thus L is injective.

Thus L is surjective. ■

Iterated slice integration of $\int_{\partial B} \frac{1 - \|x\|^2}{\|x - \zeta\|^n} f(\zeta) d\sigma(\zeta)$:

Let B_n denote the open unit ball in \mathbf{R}^n .

Let σ_n denote normalized surface area measure on ∂B_n .

Let V_n denote volume measure on B_n .

Theorem Let f be a Borel measurable, integrable function on ∂B_n . If $1 \leq k < n$, then

$$\int_{\partial B_n} f d\sigma_n = \frac{k}{n} \frac{V(B_k)}{V(B_n)} \int_{B_{n-k}} (1 - |x|^2)^{\frac{k-2}{2}} \int_{\partial B_k} f(x, \sqrt{1 - |x|^2} \zeta) d\sigma_k(\zeta) dV_{n-k}(x).$$

Power series expansion of the Poisson kernel:

$$\int_{\partial B} \frac{1 - \|x\|^2}{\|x - \zeta\|^n} f(\zeta) d\sigma(\zeta)$$

For u defined on some subset of \mathbf{R}^n , the *Kelvin transform* of u is the function $\mathcal{K}[u]$ defined by

$$\mathcal{K}[u](x) = \|x\|^{2-n} u\left(\frac{x}{\|x\|^2}\right).$$

Theorem (Kelvin): u is harmonic if and only if $\mathcal{K}[u]$ is harmonic.

Theorem (Axler and Ramey, 1995): The Poisson integral of a polynomial f can be computed rapidly from

$$\mathcal{K}[D_f \|x\|^{2-n}].$$



Lord Kelvin (1824-1907)

How can we compute the solution to the Dirichlet problem for the unit ball?

- 1 Iterated slice integration.
- 2 Power series expansion of the Poisson kernel.
- 3 Kelvin transform.

None of these methods work for ellipsoids!

Theorem (Axler, Gorkin, and Voss, 2004): The solution to the Dirichlet problem for ellipsoids can be computed reasonably fast by repeated differentiation.

Link for software

The *Mathematica* package that implements these algorithms is available (without charge) at the following link:

http://axler.net/HFT_Math.html



Best wishes to John Conway!!!